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#### Citation for published version:

Del Debbio, L & Diakonov, D 2002, 'Center vortices and k-strings', *Physics Letters B*, vol. 544, no. 1-2, pp. 202–206. [https://doi.org/10.1016/S0370-2693\(02\)02473-5](https://doi.org/10.1016/S0370-2693(02)02473-5)

#### Digital Object Identifier (DOI):

[10.1016/S0370-2693\(02\)02473-5](https://doi.org/10.1016/S0370-2693(02)02473-5)

#### Link:

[Link to publication record in Edinburgh Research Explorer](#)

#### Published In:

Physics Letters B

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# Center vortices and $k$ -strings

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## Abstract

The vortex contribution to the  $k$ -string tensions is computed for  $SU(N)$  gauge theories. We deduce the surface densities needed to reproduce the sine scaling and the Casimir scaling formulae, recently obtained from numerical simulations on the lattice. We find that such densities need to grow linearly in  $N$ , which in turn suggests that the vortex scenario can hardly reproduce the physics of confinement at large  $N$ .

## 1 Introduction

In a pure glue  $SU(N)$  gauge theory one expects an asymptotic area law for large Wilson loops in the fundamental representation of the group and, in general, in all representations with nonzero ' $N$ -ality'. A representation is said to have zero  $N$ -ality if it transforms trivially under the center of the group  $Z_N$ , or, equivalently, if it is contained in the direct product of some number of adjoint representations. Wilson loops in  $N$ -ality-zero representations cannot exhibit the asymptotic area law since the static source in such representations can be screened by gluons in the adjoint representation.

From the confinement point of view, a special role is played by representations given by rank- $k$  antisymmetric tensors, with  $k = 1, \dots, (N - 1)$ . The value  $k = 1$  corresponds to the fundamental representation whereas  $k = N - 1$  corresponds to the representation conjugate to the fundamental. In general, the rank- $(N - p)$  antisymmetric representation is complex conjugate to the rank- $p$  one, and has the same dimension. The dimension of the rank- $k$  antisymmetric representation is

$$d(k, N) = C_N^k = \frac{N!}{k!(N - k)!}. \quad (1)$$

Normalizing the generators in the rank- $k$  representation as

$$[T^a T^b] = i f^{abc} T^c, \quad f^{abc} f^{dbc} = \delta^{ad} N, \quad (2)$$

one finds the eigenvalue of the quadratic Casimir operator to be

$$T^a T^a = \frac{N+1}{2N} k(N-k) \mathbf{1}. \quad (3)$$

None of the  $k$ -representations are contained in the direct product of adjoint representations, so they have a nonzero  $N$ -ality. Furthermore, *any*  $N$ -ality nonzero representation is contained in a direct product of a certain rank- $k$  antisymmetric representation and some number of adjoint representations. In other words, *all* representations are equivalent, from the confinement point of view, to some rank- $k$  antisymmetric representation. (If  $k=0$ , it is a  $N$ -ality-zero representation.) Therefore, all Wilson loops fall into  $N$  classes according to what  $k$ -representation is the given source equivalent to, modulo adjoint representations. Correspondingly, there are only  $N$  string tensions  $\sigma(k, N)$ ,  $k=1\dots N$ , to characterize confinement in all possible representations of the trial source, with  $\sigma(N, N)=0$  and  $\sigma(1, N)=\sigma_{\text{fund}}(N)$ .  $\sigma(k, N)$  can be interpreted as the tension of a string connecting  $k$  quarks to  $k$  antiquarks. Clearly, such a configuration is stable only if the energy is less than the energy of  $k$  fundamental strings connecting the same color sources. The symmetry under charge conjugation, already mentioned above, implies that  $\sigma(k, N)=\sigma(N-k, N)$ , leaving only  $[N/2]$  independent quantities. Finally, it is worthwhile to remember that exact factorization in the large- $N$  limit requires:

$$R(k, N) = \frac{\sigma(k, N)}{\sigma(1, N)} \xrightarrow{N \rightarrow \infty} k. \quad (4)$$

In this work, we examine the vortex contribution to rank- $k$  antisymmetric Wilson loops, which leads to a prediction for  $\sigma(k, N)$  as a function of the density of  $l$ -vortices carrying flux  $l$ . By comparing this prediction with recent numerical results for the  $k$ -string tensions [1, 2], we are able to actually compute the vortex densities that are needed to fit the results from Monte Carlo simulations.

## 2 Wilson loops in $k$ -representations

Let the Wilson loop in the fundamental representation be some unitary matrix  $U_\beta^\alpha$ . In  $k=2$  antisymmetric representation the same Wilson loop is then  $U_{[\beta_1 \beta_2]}^{[\alpha_1 \alpha_2]} = \frac{1}{2} (U_{\beta_1}^{\alpha_1} U_{\beta_2}^{\alpha_2} - U_{\beta_2}^{\alpha_1} U_{\beta_1}^{\alpha_2})$ . For a general rank- $k$  representation one has to antisymmetrize the product of  $k$  fundamental unitary matrices  $U_\beta^\alpha$ . We introduce the trace of the Wilson loop  $W(k, N)$  in the  $k$ -representation, normalized in such a way that all  $W$ 's are unity at  $U = \mathbf{1}$ . We have:

$$\begin{aligned} W(1, N) &= \frac{1}{N} \text{Tr } U, \\ W(2, N) &= \frac{2}{N(N-1)} ((\text{Tr } U)^2 - \text{Tr } U^2), \\ W(3, N) &= \frac{1}{d(3, N)} ((\text{Tr } U)^3 - 3 \text{Tr } U^2 \text{Tr } U + 3 \text{Tr } U^3), \end{aligned} \quad (5)$$

and so on.

We now turn to the center vortices; there are also  $(N - 1)$  types of them and we shall name them  $l$ -vortices,  $l = 1, \dots, (N - 1)$ . By definition of an  $l$ -vortex, the Wilson loop in the fundamental representation of the  $SU(N)$  gauge group, winding around the  $l$ -vortex in the transverse plane is gauge-equivalent to a diagonal unitary matrix being a non-trivial element of the group center, as the radius of the Wilson loop tends to infinity:

$$U_\beta^\alpha = \text{P exp } i \oint A_\mu dx^\mu \rightarrow \begin{pmatrix} z_N^l & & \\ & \ddots & \\ & & z_N^l \end{pmatrix} \in Z(N), \quad z_N^l \equiv e^{2\pi i l/N}. \quad (6)$$

From Eq. (5) we immediately find the corresponding traces of Wilson loops in the  $k$ -representation:

$$W(k, N) = z_N^{kl} = e^{2\pi i kl/N}. \quad (7)$$

### 3 Area law for $k$ -loops from $l$ -vortices

Let us consider the following simplest scenario: The Yang–Mills vacuum is populated by random center  $l$ -vortices. Let us assume the vortices are statistically independent. The probability that  $n_l$  vortices pierce a given Wilson loop is then given by the Poisson distribution,

$$P_{n_l} = \frac{\bar{n}_l^{n_l}}{n_l!} e^{-\bar{n}_l}, \quad \sum_{n_l} P_{n_l} = 1, \quad (8)$$

where the average number of  $l$ -vortices going through a surface spanned over the Wilson loop is

$$\bar{n}_l = \rho(l, N) \cdot \text{Area} \quad (9)$$

where  $\rho(l, N)$  is the average density of  $l$ -vortices in the  $SU(N)$  gauge theory, piercing any two-dimensional plane; its dimension is  $1/\text{cm}^2$ .

According to eq. (7) each  $l$ -vortex contributes to the  $k$ -loop a factor  $z_N^{kl}$ . Assuming the Poisson distribution of the vortices (8) we get for the average  $k$ -loop:

$$\begin{aligned} W(k, N) &= \prod_{l=1}^{N-1} \sum_{n_l=0}^{\infty} P_{n_l} (z_N^{kl})^{n_l} \\ &= \prod_{l=1}^{N-1} \exp(-\bar{n}_l + \bar{n}_l z_N^{kl}) \\ &= \exp \left[ -\text{Area} \sum_{l=1}^{N-1} \rho(l, N) (1 - z_N^{kl}) \right]. \end{aligned} \quad (10)$$

This is the area law with the string tension

$$\sigma(k, N) = \sum_{l=1}^{N-1} \rho(l, N) (1 - e^{2\pi i kl/N}). \quad (11)$$

Using the fact that  $\rho(l, N) = \rho(N - l, N)$ , the above equation can be rewritten as

$$\sigma(k, N) = \sum_{l=1}^{[N/2]} 2\rho(l, N) \left(1 - \cos \frac{2\pi kl}{N}\right) \quad (12)$$

if  $N$  is odd, while for even  $N$  one gets:

$$\sigma(k, N) = \sum_{l=1}^{N/2-1} 2\rho(l, N) \left(1 - \cos \frac{2\pi kl}{N}\right) + \rho\left(\frac{N}{2}, N\right) \left(1 - (-1)^k\right). \quad (13)$$

In both cases, one obtains explicitly a positive, real number for the string tension. In particular, for the  $SU(2)$  gauge group there is only one type of center vortex with  $l = 1$  and  $z_2 = -1$ , and only one non-trivial representation with  $k = 1$ , the fundamental one. In this case we recover the well-known result:

$$\sigma(1, 2) = 2\rho(1, 2). \quad (14)$$

In  $SU(3)$  one can have  $k, l = 1, 2$ ; however  $k = 2$  is the antitriplet which is conjugate to the fundamental representation, while  $l = 2$  corresponds to the vortices conjugate to those with  $l = 1$  so that their densities must be equal,  $\rho(1, 3) = \rho(2, 3)$ . In this case we get from eq. (11) only one non-trivial string tension:

$$\sigma(1, 3) = \sigma(2, 3) = 3\rho(1, 3) = 3\rho(2, 3). \quad (15)$$

Starting from  $SU(4)$  there are different types of  $l$ -vortices and correspondingly different string tensions for  $k$ -loops. One could naively assume that the vortex density is independent of  $l$ , i.e.  $\rho(l, N) = \rho_N$ . This oversimplified scenario can be immediately ruled out by noting that it leads to  $k$ -string tensions that are independent of  $k$  and proportional to  $N$ , namely  $\sigma(k, N) = \rho_N N$ , for each  $k$ . The inconsistency of this result with large- $N$  factorization immediately points toward more refined scenarios. The next Section is devoted to the determination of the vortex densities needed to reproduce the  $k$ -string tensions recently computed by numerical simulations.

## 4 Two scenarios

Two types of behaviour of the  $k$ -strings have been discussed in the literature: the “sine” regime [3, 4, 5],

$$\sigma_{\sin}(k, N) = \sigma_1 \frac{\sin \frac{\pi k}{N}}{\sin \frac{\pi}{N}}, \quad (16)$$

and the “Casimir” regime [6],

$$\sigma_{\text{cas}}(k, N) = \sigma_1 \frac{k(N - k)}{N - 1}. \quad (17)$$

Recent numerical simulations show that the “sine” scaling hypothesis provides an accurate description of the  $k$ -string spectrum to the level of a few percent [2]. On the other hand, even

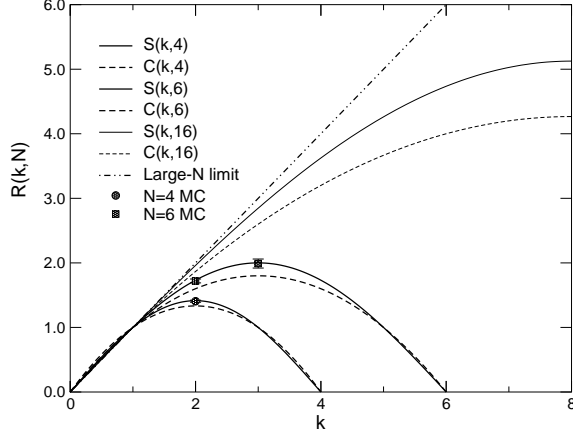


Figure 1: Ratios of  $k$ -string tensions to the fundamental ones for gauge groups SU(4), SU(6), SU(16) and SU( $\infty$ ), in the “sine” and “Casimir” scenarios. Also shown are lattice-measured values for SU(4) and SU(6) [2].

the Casimir scaling formula turns out to be a good approximation at the level of 10% [1, 2]. In both cases  $\sigma_1$  is the string tension in the fundamental representation of the SU( $N$ ) group; it may depend somewhat on  $N$  but asymptotically it is believed that  $\sigma_1 = O(N^0)$ , i.e. that it is independent of  $N$ . The two formulae are plotted in Fig. 1.

The two regimes for the string tension correspond to two different types of behaviour for the two-dimensional center vortices densities  $\rho(l, N)$ , as functions of their ‘flux number’  $l$ . To find  $\rho(l, N)$ , one needs to equate the general expression (11) to the appropriate string tension, eq. (16) or eq. (17). Eq. (11) is in fact a Fourier series, and it is not difficult to calculate  $\rho(l, N)$  from the inverse Fourier transformation. We find:

$$\rho_{\text{sin}}(l, N) = \sigma_1 \frac{1}{N} \frac{1}{\cos \frac{\pi}{N} - \cos \frac{2\pi l}{N}}, \quad (18)$$

$$\rho_{\text{cas}}(l, N) = \sigma_1 \frac{1}{2(N-1)} \frac{1}{\sin^2 \frac{\pi l}{N}}. \quad (19)$$

The two formulae are plotted in Fig. 2. In both cases one reproduces the relation  $\rho(l, N) = \rho(N-l, N)$ , as it should be since  $(N-l)$ -vortices are complex conjugate to the  $l$ -vortices. The asymptotics of eqs. (18,19) at large  $N$  and fixed  $l$  are:

$$\rho_{\text{sin}}(l, N) \rightarrow \sigma_1 \left[ \frac{2N}{\pi^2(4l^2-1)} + \frac{1}{6N} \frac{4l^2+1}{4l^2-1} + O\left(\frac{1}{N^2}\right) \right], \quad (20)$$

$$\rho_{\text{cas}}(l, N) \rightarrow \sigma_1 \left[ \frac{N^2}{2(N-1)\pi^2 l^2} + \frac{1}{6(N-1)} + O\left(\frac{1}{N^2}\right) \right]. \quad (21)$$

Since the string tension in the fundamental representation  $\sigma_1 = O(N^0)$ , we see that both regimes above require the two-dimensional densities of vortices to grow linearly with  $N$ . At the same time, the transverse sizes of vortices are stable in  $N$ , as it follows from the dimensional analysis of ref. [7]. It means that at large  $N$  center vortices are inevitably strongly overlapping, and the notion of vortices loses sense.

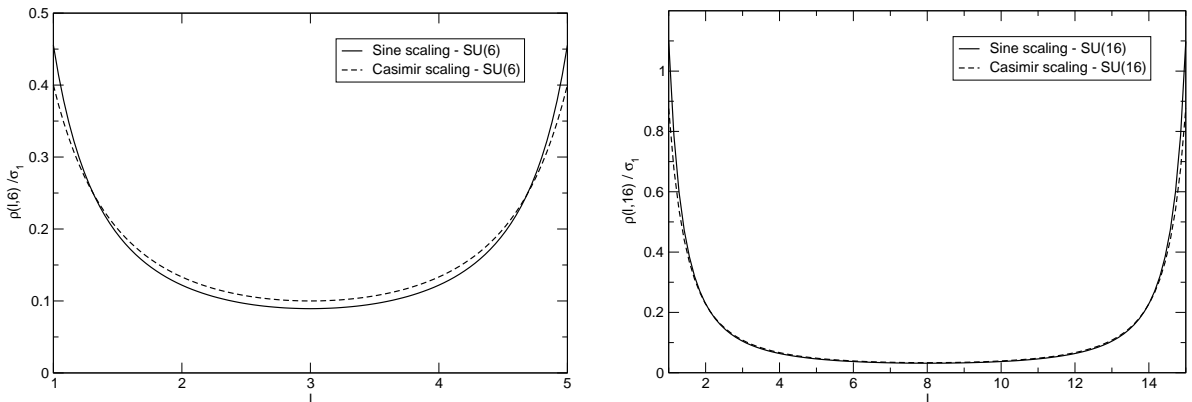


Figure 2: Densities of center vortices assuming the “sine” (solid) or “Casimir” (dashed) scenarios for the  $k$ -string tensions. The gauge groups are SU(6) (left) and SU(16) (right). In both scenarios, the vortex densities increase and become peaked around small values of  $l$  as the number of colors grows large.

In principle, one can imagine a regime where the density of vortices does not grow with  $N$ . For example, one can assume, as a matter of an exercise, that only maximal-flux  $l$ -vortices exist, with  $l \approx \frac{1}{2}N$ . Apart from being a rather unnatural hypothesis, it is in contradiction with the  $k$ -string tensions measured in lattice simulations [1, 2].

## 5 Conclusions

From the viewpoint of the confinement behaviour, sources in various representations of the SU( $N$ ) gauge group fall into  $N$  classes with the same string tension  $\sigma(k, N)$  for all representations belonging to the same  $k$ -class. The value  $k = 0$  (or equivalently  $k = N$  since  $\sigma(k, N) = \sigma(N - k, N)$ ) labels  $N$ -ality zero representations *i.e.* adjoint and those which arise in a direct product of any number of adjoint representations. The string tension here is zero as sources in such representations are screened by gluons. Sources in  $N$ -ality nonzero representations can be partially screened: the representation belongs to the class of the rank- $k$  antisymmetric tensor if it is found in the direct product of that representation with some number of adjoint ones. Therefore, the  $k$ -string tensions  $\sigma(k, N)$  are the only fundamental ones which can appear in the asymptotic area law. The dynamics of confinement is encoded in these numbers.

Assuming that confinement is driven by center vortices populating the Yang–Mills vacuum, one derives relations between the surface densities of  $l$ -vortices piercing any given two-dimensional plane, where  $l = 1, \dots, (N - 1)$  is the flux of an SU( $N$ ) center vortex. In fact, these densities  $\rho(l, N)$  are Fourier coefficients of the  $k$ -string tensions  $\sigma(k, N)$ . We have found the densities needed to reproduce two popular regimes for  $k$ -strings: the “sine” and the “Casimir” regimes, see Figs. 1,2. In both cases we find that the density of vortices should rise linearly with  $N$ . It means that the vortex scenario of confinement can hardly be stable in  $N$ : at large enough  $N$  vortices have to overlap, and the vortex language probably becomes senseless.

## Acknowledgements

LDD thanks the Niels Bohr Institute for hospitality in the early stages of this work.

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